Lecture 19.

More on normal Random Variables.

Recall from yesterday that $X = N(\mu, \sigma^2)$ is normally distributed with parameters μ, σ^2 $(\sigma > 0)$ iff X has a probability density function given by

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}.$$

We showed yesterday that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, even though $f_X(x)$ has no antiderivative.

<u>Prop:</u> Let $X = N(\mu, \sigma^2)$ be a normal random var. For any $a, b \in \mathbb{R}$, $a \neq 0$, $Y = aX + b$ is normal.

<u>Pf:</u> Let $F_Y(x)$ be the cumulative distribution function of Y. Then

$$\begin{aligned} F_Y(x) &:= P(Y \le x) = P(aX + b \le x) \\ &= P\left(X \le \frac{x-b}{a}\right) \\ &= F_X\left(\frac{x-b}{a}\right). \end{aligned}$$

Since $$F_Y(x) = F_X\left(\frac{x-b}{a}\right)$$
we can differentiate both sides to get
$$f_Y(x) = f_X\left(\frac{x-b}{a}\right)\frac{1}{a}.$$
Since $f_X(x) = \dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\;}$, we have

$$f_Y(x) = \frac{e^{-\left(\frac{x-b}{a}-\mu\right)^2/2\sigma^2}}{a\sigma\sqrt{2\pi}}$$

$$= \frac{e^{-(x-b-a\mu)^2/2\sigma^2a^2}}{a\sigma\sqrt{2\pi}}$$

$$= \frac{e^{-(x-(a\mu+b))^2/2(\sigma a)^2}}{\sigma a\sqrt{2\pi}}$$

So $Y = N(a\mu+b, (\sigma a)^2)$. ///.

<u>Definition</u>: $Z = N(0, 1)$ is called the <u>standard normal random variable</u>.

<u>Corollary</u>: Let $X = N(\mu, \sigma^2)$. Then $\dfrac{X-\mu}{\sigma}$ is standard.

<u>Pf</u>: $Y = \dfrac{X-\mu}{\sigma} = \overset{a}{(1/\sigma)}X + \overset{b}{\left(\frac{-\mu}{\sigma}\right)}$.

So $Y = N\left(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \left(\sigma\cdot\frac{1}{\sigma}\right)^2\right) = N(0, 1)$. $\square$

See text for verification that, for $Z = N(0,1)$,

$$E[Z] = 0$$
$$Var(Z) = 1.$$

It follows that if $X = \mu + \sigma Z = N(\mu, \sigma^2)$ is a normal random var, then.

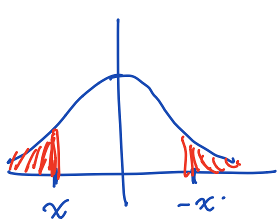
$$E[X] = \mu.$$
$$Var(X) = \sigma^2.$$

For $Z = N(0,1)$ we denote the CDF of Z as

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

<u>Claim</u>: $\forall x,\ \Phi(-x) = 1 - \Phi(x).$

<u>Pf</u>: Since $f_Z(x)$ is symmetric about 0, i.e. $f_Z(x) = f_Z(-x)$, we have



$$P(Z \le x) = P(Z > -x) \quad \forall x$$
$$= 1 - P(Z \le -x)$$

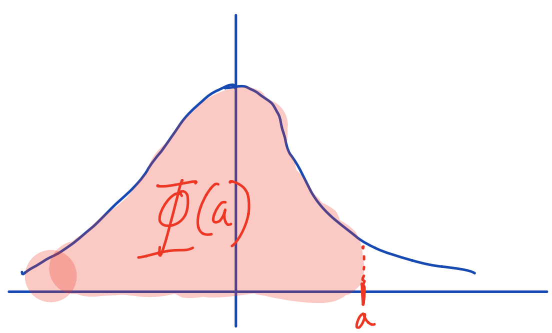
so $P(Z \le x) + P(Z \le -x) = 1.$

i.e. $\Phi(x) + \Phi(-x) = 1.$ ✓

Suppose now that $X \sim N(\mu, \sigma^2)$ is a normal RV. For any $x \in \mathbb{R}$, we have that

$$F_X(x) = P(X \leq x)$$
$$= P\Big(\underbrace{\frac{X-\mu}{\sigma}}_{N(0,1)} \leq \frac{x-\mu}{\sigma}\Big)$$
$$= P\Big(Z \leq \frac{x-\mu}{\sigma}\Big).$$
$$= \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

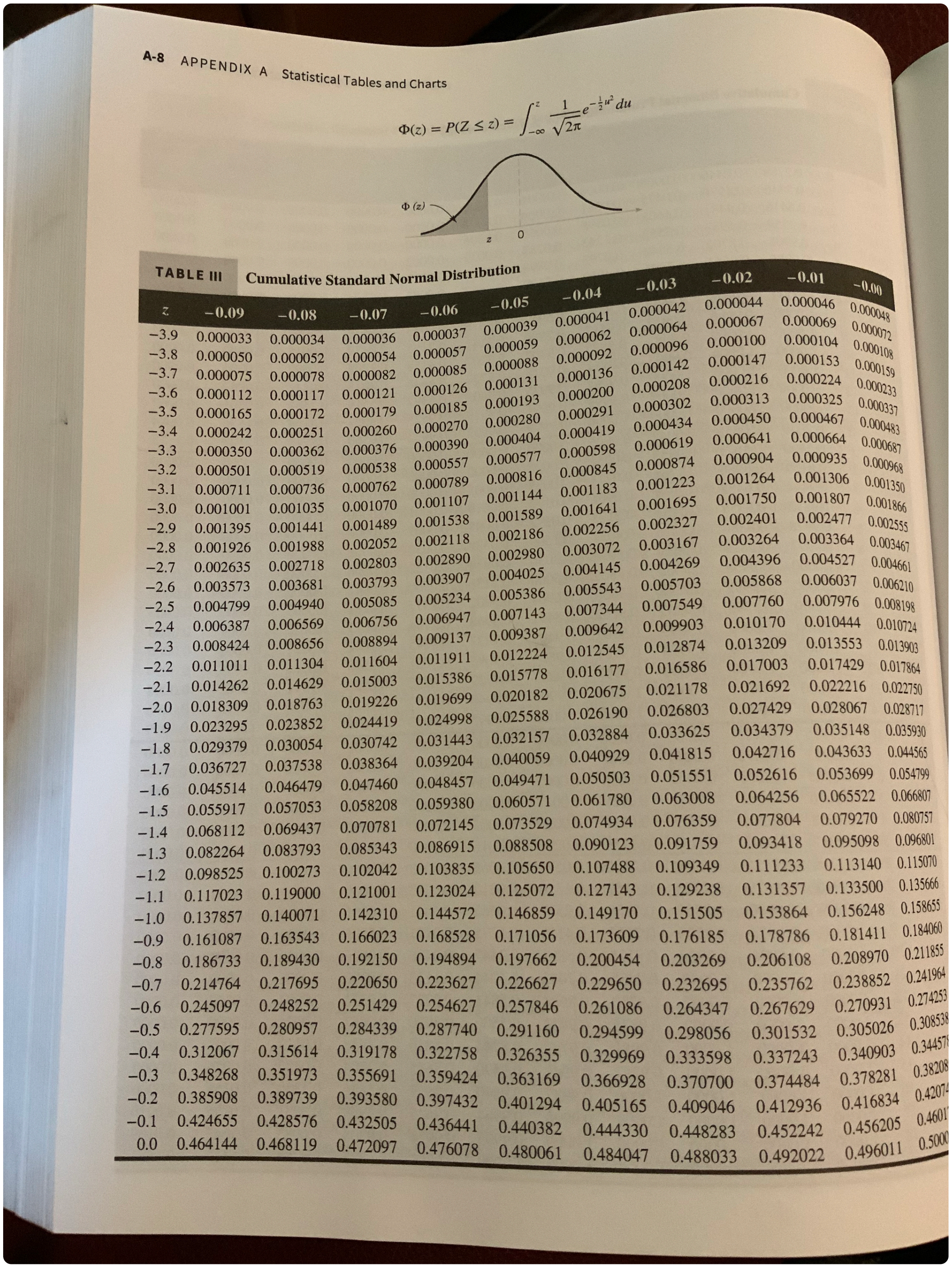
Therefore, even though f_X has no antiderivative, we can still compute/approximate $F_X(a)$ using "Z-score tables.



Z	b
a	$\Phi(a+b)$

Ex:

	0.00	0.01	0.02	0.03	
0.0	0.5	0.504			
0.1	0.539				
0.2					
0.3					
0.4					



A-8 APPENDIX A Statistical Tables and Charts

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

Φ (z)

z 0

TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.34457
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.38208
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.4207
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.4601
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.5000

Example of a table of Z-scores

Note: The Casio Fx 991ms can compute Z-scores for you:

See the following video tutorial that I just googled:

https://youtu.be/Ugdngb1jy7s

Example: Suppose $X = N(3, 9)$. Find $P(2 \le X \le 5)$.

Recall: For any continuous RV, X,

$$P(a \le X \le b) = F_X(b) - F_X(a).$$

So

$$P(2 \le X \le 5) = F_X(5) - F_X(2)$$

$$= \Phi\left(\frac{5-3}{3}\right) - \Phi\left(\frac{2-3}{3}\right)$$

$\sigma = 3 = \sqrt{9}$

$$= \Phi\left(\frac{2}{3}\right) - \Phi\left(\frac{-1}{3}\right)$$

$$= \underbrace{\Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right)}_{\text{look up/use calculator}}.$$

$$\approx 0.3779.$$

Ex: Let $X = N(3, 9)$. Find a symmetric interval I around $\mu = 3$ such that $P(X \in I) = 0.99$.

Solution: We want to find some value $t \geq 0$ such that $P(X \in [3-t, 3+t]) = 0.99$.

i.e. $P(3-t \leq X \leq 3+t) = 0.99$.

So

$$0.99 = F_X(3+t) - F_X(3-t)$$
$$= \Phi\left(\frac{3+t-3}{3}\right) - \Phi\left(\frac{3-t-3}{3}\right)$$
$$= \Phi\left(\frac{t}{3}\right) - \Phi\left(-\frac{t}{3}\right)$$
$$= \Phi\left(\frac{t}{3}\right) - \left(1 - \Phi\left(\frac{t}{3}\right)\right)$$
$$0.99 = 2\Phi\left(\frac{t}{3}\right) - 1.$$

2.575.

So $\Phi\left(\frac{t}{3}\right) = \frac{1.99}{2} = 0.995$.

Consulting the z-score table,

$$\Phi(2.575) \approx 0.995,$$

So set $\frac{t}{3} = 2.575 \Rightarrow \boxed{t = 7.725.}$

So $I = [-4.725, 10.725]$.

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